PMM U.S.S.R.,Vol.49,No.6,pp.725-732,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON THE INTEGRABLE CASE OF A RIEMANN-HILBERT BOUNDARY VALUE PROBLEM FOR TWO FUNCTIONS AND THE SOLUTIONS OF CERTAIN MIXED PROBLEMS FOR A COMPOSITE ELASTIC PLANE*

I.V. SIMONOV

A method for solving the Riemann-Hilbert boundary value problem with piecewise-constant coefficients is generalized /1/. It is shown that the following static problems of a composite elastic plane with three kinds of connection conditions allow of exact solutions: 1) the splicing line is weakened by a system of loaded slots and a transverse shear crack or the edges of one of the slots are partially contacting, or one of the slots is cleaved by a rigid insert; 2) the splicing line is reinforced by a system of thin rigid inclusions and there is one arbitrarily located delamination zone; 3) the elastic half-planes are contacting (with slip) on a certain section of their boundaries, and mixed boundary conditions in the displacements and stresses are given on the rest of the boundaries.

In the general case the Riemann-Hilbert boundary value problem for many functions reduces to the problem of a linear conjugation, and then to Fredholm integral Eqs./2/. Closed solutions are obtained in certain special cases /3—5/. For applications we mention the papers /6, 7/, where problems are considered concerning slits at the interface of two elastic media with two kinds of physical boundary conditions taken into account simultaneously.

1. We consider the following boundary value problem of the theory of functions of a complex variable. It is required to find the vector function $\Phi(z) = (\Phi_1(z), \Phi_2(z))$ which is analytic in the upper $z = x + iy$ half-plane, vanishes at infinity, and is continuously continuable on the real axis $y = +0$ except, perhaps, at the points $a_k, b_k, \pm 1$ ($|a_k|, |b_k| \geqslant 1$, $k = 1, \ldots, N$, $2 \leqslant N < \infty$) in whose neighbourhoods the following estimates hold:

$$|\Phi| < C' |z - a|^{\rho}, \ C > 0, \ 0 \leqslant \rho < 1 \tag{1.1}$$

where a is any of the points $a_k, b_k, \pm 1$, for the following boundary conditions:

$$\begin{gathered}\operatorname{Im}(D\Phi)(x) = f(x) \quad \text{on} \quad l = \{z = x + i0, \ x \neq a\} \\ D = D_m, \ x \in l_m \ (m = 0, 1, 2); \ f(x) \in H_\varepsilon \ \ (f(x) \to 0, \quad |x| \to \infty) \\ l_0 =]-1, 1[, \ \ l_1 = \{a_k b_k\}, \ \ l_2 = l - l_0 - l_1\end{gathered} \tag{1.2}$$

where D is a non-singular piecewise-constant matrix.

Therefore, the solution of the Riemann-Hilbert problem (1.2) is sought in the class of functions h_0 /2/ defined by the estimates (1.1) and the condition at infinity. The domain boundary is separated into a system of intervals l_1, l_2 and the isolated interval l_0. Without loss of generality, we set $f(x) = 0$, $x \in l_0$, and $D_0 = E$ where E is the unit matrix.

In the trivial case of triangular matrices D_1 and D_2 the vector problem (1.2) splits at once into a chain of sequentially solvable scalar problems. If one of the matrices D_1, D_2 contains just imaginary elements, then equality of two (out of the three) matrices D can be achieved at the beginning by a linear transformation of the desired function, and we then arrive at a conjugate problem allowing of splitting by reduction of the matrix coefficient of the problem to a diagonal or triangular form.

The case is examined below when the upper rows of the matrices D_1 and D_2 are real, while the lower rows are imaginary numbers. The linear substitution $\Phi^\circ = D_2^\circ \Phi$, where D_2° is a real matrix formed from elements of the matrix D_2 (the upper rows of these matrices coincide while the lower rows differ only by the factor i), reduces to problem (1.2) in which

$$D_0 = E, D_1 = \begin{Vmatrix} d_{11} & d_{12} \\ id_{21} & id_{22} \end{Vmatrix}, \quad D_2 = \begin{Vmatrix} 1 & 0 \\ 0 & i \end{Vmatrix} \tag{1.3}$$

$$d_{mj} = \bar{d}_{mj} \neq 0, \quad m, j = 1, 2$$

The bar denotes the complex conjugate and the superscript $^\circ$ on the vector function is omitted.

The problem remains essentially connected, the matrix D_1 contains no zero elements and

*Prikl.Matem.Mekhan.,49,6,951-960,1985

all three matrices are different.

We continue the function Φ analytically through the interval l_0. We transfer the z plane with the slits $y = \pm 0$, $|x| \geqslant 1$ by the conformal transformation $\omega = \xi + i\eta = z + \sqrt{z^2 - 1}$ into the upper ω half-plane ($z = {}^1/_2(\omega + \omega^{-1})$ is the Zhukovskii transformation). The correspondence between the boundaries and the points is as follows. The upper edges of the slits $l_1^+ + l_2^+$ transfer into the rays $L_1' + L_2' = \{|\xi| < 1, \ \xi \neq A_k, B_k\}$ on the real axis of the ω-plane with deleted points $A_k, B_k \Leftrightarrow a_k, b_k$; the intervals $L_1'' + L_2'' = \{|\xi| < 1, \ \xi \neq A_k^{-1}, B_k^{-1}\}$ correspond to the lower edges of the slits $l_1^- + l_2^-$. The segment $[-1, 1]$ transfers into a unit semicircle located in the upper half-plane ω (the points $z = \pm 1$ remain fixed) so that the upper (lower) z half-plane transfers into the exterior (interior) of this semicircle. We shall also keep in mind the following asmyptotic forms

$$\omega(z) \sim 2z, \quad y > 0; \quad \omega(z) \sim (2z)^{-1}, \quad y < 0 \quad (z \to \infty) \tag{1.4}$$

For $\eta = 0$ it follows from (1.2) (we do not alter the notation of the functions)

$$\operatorname{Im}(D\Phi) = f(\xi), \quad |\xi| > 1; \quad \operatorname{Im}(\bar{D}\Phi) = -f(\xi), \quad |\xi| < 1$$
$$D = D_m, \quad \xi \in L_m = L_m' + L_m'', \quad m = 1, 2$$

The case (1.3) is a non-trivial modification of the problem when the conjugation sign in the boundary condition for $|\xi| < 1$ can be removed and the problem can be reduced to the form

$$\operatorname{Im}(D\Phi) = g(\xi), \quad |\xi| < \infty \tag{1.5}$$
$$g = (g_1, g_2) = f(\xi), \quad |\xi| > 1; \quad g_m(\xi) = (-1)^m g_m(1/\xi)$$
$$|\xi| < 1$$

$$\Phi(\omega) = \overline{\Phi(1/\bar{\omega})}, \quad \eta \geqslant 0 \tag{1.6}$$

It is remarkable that the matrix coefficient D in (1.5) takes just two values on the real axis, the boundary condition on l_0 transfer into (1.6), i.e., in the class of additional conditions of the Riemann-Hilbert problem (1.5).

The piecewise-holomorphic vector

$$Y(\omega) = D_2\Phi(\omega), \ \eta \geqslant 0; \ Y(\omega) = \overline{Y(\bar{\omega})}, \ \eta \leqslant 0 \tag{1.7}$$

on the jump lines $\eta = 0$ should satisfy the conjugate conditions resulting from (1.5) and (1.7)

$$Y^+ = D'Y^- + 2iBg(L_1), \ Y^+ = Y^- + 2ig(L_2) \tag{1.8}$$
$$D' = \begin{Vmatrix} d_0 & id_1 \\ id_2 & d_0 \end{Vmatrix}, \quad d_0 = \frac{1+\alpha}{1-\alpha}, \quad d_1 = \frac{2d_{12}}{d_{11}(1-\alpha)}$$
$$d_2 = \frac{-2d_{21}}{d_{22}(1-\alpha)}, \quad \alpha = \frac{d_{12}d_{21}}{d_{11}d_{22}}, \quad B = D_2D_1^{-1}$$

where the plus (minus) superscript denotes shrinkage from above (below) on the $\eta = 0$ axis. By virtue of the non-degeneracy of the matrix D_1 and the condition $d_{mj} \neq 0$ we have $\alpha \neq 0, 1, \infty$. The linear substitution $Y = TW$, where T is a diagonalizing matrix, results in a split conjugate problem for the piecewise-holomorphic vector $W = (W_1, W_2)$

$$W^+ = \Lambda W^- + 2iW^\circ(L_1), \ W^+ = W^- + 2iW^\circ(L_2) \tag{1.9}$$
$$\Lambda = T^{-1}D'T = \begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{Vmatrix}, \quad T = \begin{Vmatrix} 1 & 1 \\ t & -t \end{Vmatrix}, \quad t = \begin{cases} ib, & \alpha > 0 \\ b, & \alpha < 0 \end{cases}$$
$$b = \sqrt{\left|\frac{d_2}{d_1}\right|}, \quad \lambda_1 = \lambda_2^{-1} = \frac{1 - s\sqrt{\alpha}}{1 + s\sqrt{\alpha}}, \quad \alpha > 0, \quad s = \operatorname{sgn}\left(\frac{d_{21}}{d_{22}}\right)$$
$$\lambda_1 = \bar{\lambda}_2 = \frac{1 - is\sqrt{-\alpha}}{1 + is\sqrt{-\alpha}}, \quad \alpha < 0$$
$$W^\circ(\xi) = T^{-1}Bg(\xi)(L_1), \ W^\circ(\xi) = T^{-1}g(\xi) \ (L_2)$$

$$D_2^{-1}TW(\omega) = \overline{D_2^{-1}TW(1/\bar{\omega})}, \quad TW(\omega) = \overline{TW(\bar{\omega})} \tag{1.10}$$

We obtain the total equivalence of problem (1.9), (1.10) to the initial formulation (1.1), (1.2) by subjecting the behaviour of the function $W(\omega)$ at the singularities to estimates that follow from (1.1) and the definitions of this section, and by requiring the vanishing at infinity according to (1.4). It follows from (1.3) that the indices of the singularities of the canonical solutions /2, 8/ near the points $z = \pm 1$ should be $(0, -{}^1/_2)$. In other words, the matrices D are chosen such that the discontinuities in the boundary conditions at the points ± 1 will generate just root singularities in the function $\Phi(z)$. Then the functions $\Phi(\omega), Y(\omega), W(\omega)$ have a simple pole at the points $\omega = \pm 1$. Estimates of the behaviour of these functions near the singular points $A_k^{\pm 1}, B_k^{\pm 1} \neq \pm 1$ are analogous to (1.1); the behaviour of $W(\omega)$ at zero is controlled by the first equation in (1.10). The class of analytic functions with the above-mentioned behaviour near the points $\omega = 0, \infty, \pm 1, A_k^{\pm 1}, B_k^{\pm 1}$ and a discontinuity on $\eta = 0$ will be denoted by H_W.

Eqs.(1.9) are conditions for two independent scalar linear conjugate problems, and the matrix Λ is daigonal. We seek the general solution of problem (1.9) in the class H_W in the form of the sum of the particular solution of the inhomogeneous problem and the general solution of the corresponding conjugate homogeneous problem /8/

$$W_m(\omega) = F_m(\omega) G_m(\omega) I_m(\omega) + F_m^\circ(\omega) G_m^\circ(\omega) R_m(\omega) \tag{1.11}$$

$$I_m = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{W_m^\circ(t)\,dt}{F_m^+(t) G_m^+(t)(t-\omega)}, \quad R_m = \sum_{n_1}^{n_2} r_n^{(m)} \omega^n \quad (m = 1, 2)$$

$$\frac{G_m^+(\xi)}{G_m^-(\xi)} = \frac{G_m^{\circ+}(\xi)}{G_m^{\circ-}(\xi)} = \begin{cases} \lambda_m, & \xi \in L_1 \\ 1, & \xi \in L_2 \end{cases} \tag{1.12}$$

The auxiliary functions G here cancel the discontinuities in the coefficients of the problem at the points $A_k^{\pm 1}, B_k^{\pm 1}$, the purpose of the function F is to ensure the presence of poles in the solutions at the points ± 1, and R are rational functions in place of polynomials /8/ since the quantities G_m° will be defined as canonical solutions of the class h_0 /8/ multiplied by polynomials, $n_1 \leqslant 0, n_2 \geqslant 0$; $r_n^{(m)}$ are arbitrary complex constants.

The following properties of the free term $W^\circ(\xi)$ are needed later:

$$\begin{aligned} &W_m^\circ(\xi) = -\overline{W_m^\circ(1/\bar\xi)}, \quad |\xi| < \infty; \quad W_m^\circ(\xi) = \lambda_m \overline{W_j^\circ(\xi)} \quad (L_1) \\ &W_m^\circ(\xi) = W_j^\circ(\xi) \quad (L_2), \quad \alpha > 0 \\ &W_m^\circ(\xi) = -\overline{W_j^\circ(1/\bar\xi)}, \quad |\xi| < \infty; \quad W_m^\circ(\xi) = \lambda_m \overline{W_m^\circ(\xi)} \quad (L_1) \\ &W_m^\circ(\xi) = \overline{W_m^\circ(\xi)} \quad (L_2), \quad \alpha < 0 \quad (m, j = 1, 2; m \neq j) \end{aligned} \tag{1.13}$$

Unlike the cases studied /2/, two (and not one) additional requirements (1.10) must be satisfied. The first is the trace of the boundary condition on l_0 from (1.2), and the second is the condition of continuation of $Y(\omega)$ through the real axis. The vector components W in these relationships remain connected; however, the general principles of constructing the solution of linear conjugate problems when there is an additional condition of the type of a continuation condition /2/ transfer to the case under consideration.

Suppose $W \in H_W$ and $Y = TW$ are certain solutions of (1.9) and (1.8). The vector $Y_*(\omega) = {}^1/_2 (Y(\omega) + Y(\bar\omega))$ is also a solution of (1.18), and moreover, is subject to the rule of predetermination from (1.7). Let us form the function $\Phi_*(\omega) = D_2 Y_*(\omega)$. It can be proved that the linear combination $\Phi(\omega) = {}^1/_2 (\Phi_*(\omega) + \overline{\Phi_*(1/\bar\omega)})$ satisfies (1.5) and (1.6), i.e., all requirements of the problem. Therefore, by acting according to these general rules the solution of the original boundary value problem can be obtained, but it will be awkward. We achieve substantial simplification of the formulas by a special selection of the functions F, G, R such that the solution (1.11) will at once by subjected to the conditions (1.14) that are equivalent to (1.10)

$$\begin{aligned} &W_1(\omega) = \overline{W_2(\bar\omega)}, \; W_1(\omega) = \overline{W_1(1/\bar\omega)}, \; \alpha > 0 \\ &W_1(\omega) = \overline{W_2(1/\bar\omega)}, \; W_1(\omega) = \overline{W_1(\bar\omega)}, \; \alpha < 0 \end{aligned} \tag{1.14}$$

The desired solution will then be expressed in terms of one component $W_1(\omega)$ by means of the formulas

$$\Phi(\omega) = \left\| \begin{matrix} W_1(\omega) + \overline{W_1(\bar\omega)} \\ [W_1(\omega) - \overline{W_1(\bar\omega)}]\, b \end{matrix} \right\|, \quad \alpha > 0 \tag{1.15}$$

$$\Phi(\omega) = \left\| \begin{matrix} W_1(\omega) + \overline{W_1(1/\bar\omega)} \\ ib\,[\overline{W_1(1/\bar\omega)} - W_1(\omega)] \end{matrix} \right\|, \quad \alpha < 0$$

We substitute (1.11) into (1.14). Taking account of the properties of the eigenvalues of the matrix D' indicated in (1.9) and (1.13), we see that (1.14) and (1.15) are valid if the functions G, F and R are selected taking conditions (1.16)—(1.18) into account

$$\begin{aligned} &G_1(\omega) = \overline{G_2(\bar\omega)}, \; G_1(\omega) = \overline{G_1(1/\bar\omega)}, \; \alpha > 0 \\ &G_1(\omega) = \overline{G_2(1/\bar\omega)}, \; G_1(\omega) = \overline{G_1(\bar\omega)}, \; \alpha < 0 \end{aligned} \tag{1.16}$$

(analogous equalities hold for the function $G_m^\circ(\omega)$)

$$F_1(\omega)=\overline{F_2(\overline{\omega})},\ F_1^\circ(\omega)=\overline{F_2^\circ(\overline{\omega})}=\pm\omega^n\overline{F_1(1/\overline{\omega})} \tag{1.17}$$
$$\frac{F_1(\omega)}{\omega\overline{F_1(1/\overline{\omega})}}=\frac{F_1(\xi)}{\xi\overline{F_1(1/\xi)}}=\frac{\xi F_1(1/\xi)}{\overline{F_1(\xi)}},\quad \alpha>0$$
$$F_1(\omega)=\overline{F_1(\overline{\omega})}=\omega\overline{F_2(1/\overline{\omega})},\quad F_1^\circ(\omega)=\overline{F_1^\circ(\overline{\omega})}=\pm\omega^n F_2^\circ(1/\overline{\omega})$$
$$\alpha<0$$

$$R_1(\omega)=\overline{R_2(\overline{\omega})}=\pm\omega^{-n}\overline{R_1(1/\overline{\omega})},\quad \alpha>0 \tag{1.18}$$
$$R_1(\omega)=\overline{R_1(\overline{\omega})}=\pm\omega^{-n}\overline{R_2(1/\overline{\omega})},\quad \alpha<0$$

Here n is any integer and the signs are matched.

We construct the auxiliary functions G_m° from cofactors of the form

$$[(A_k-\omega^{-1})(\omega-B_k)]^{\gamma_m-1}[(A_k-\omega)(\omega^{-1}-B_k)]^{-\gamma_m},\quad \alpha\,\overline{\in}\,[0,1]$$

(γ_m should be substituted for γ_m-1 when $0<\alpha<1$)

$$\gamma_m=\frac{\ln\lambda_m}{2\pi}=\begin{cases}\delta_m+1/2, & \alpha\,\overline{\in}\,[0,1]\\ \delta_m, & \alpha\in\,]0,1[\end{cases} \tag{1.19}$$
$$\delta_m=\frac{\ln|\lambda_m|}{2\pi i},\quad \alpha>0\quad (m=1,2)$$
$$\delta_m=(-1)^m\left(\frac{\arg\lambda_1}{2\pi}-\frac{1}{2}\right),\quad 0<\arg\lambda_1<2\pi,\quad |\delta_m|<\frac{1}{2}$$
$$\alpha<0$$

The products of N such cofactors satisfy the factorization condition (1.12) and the symmetry condition (1.16). We draw the slits to extract the single-valued branches of the functions $G_m^\circ(\omega)$ (and $G_m(\omega)$) along $L_1=\{A_kB_k\}+\{B_k^{-1}A_k^{-1}\}$ and we fix the choice of the branch of cofactors of the form $(A_k-\omega^{\pm1})^{\gamma_m}$ and $(\omega^{\pm1}-B_k)^{\gamma_m}$ by the condition $1^{\gamma_m}=1$. The increments $\arg(A_k-\omega^{\pm1})$ and $\arg(\omega^{\pm1}-B)$ during traversal of the points $A_k^{\pm1}, B_k^{\pm1}$ from the upper to the lower edge of the slit are respectively equal to $\pm2\pi$ and $\mp2\pi$.

The functions $G_m(\omega)$ are constructed analogously. Conditions for the existence of the integrals $I_m(\omega)$ of (1.11) and the disappearance of the solution at infinity are additionally taken into account in selecting the functions $G_m(\omega)$, $F_m(\omega)$. They are consequently sometimes different from the functions $G_m^\circ(\omega)$, $F_m^\circ(\omega)$.

We have the following behaviour of the functions near the singularities $a\neq\pm1,\infty$: $\Phi(z)\sim(z-a)^{-\gamma_m}$, where the exponents γ_m defined in (1.19) take complex ($\mathrm{Re}\,\gamma_m=1/2$), imaginary, or real $(0<\gamma_m<1)$ values depending on α.

We present formulas for the auxiliary functions by making the location of the interval l_0 and the point $z=\infty$ specific relative to the subsets l_1 and l_2.

1) The section l_0 is surrounded by the intervals $\in l_2$ ($a_k\neq-\infty,1;\ b_k\neq-1,\infty;\ k=1,\ldots,N$).

Conditions (1.12), (1.16)–(1.18) realize the functions

$$G_1^\circ(\omega)=\Pi_1(\omega)\Pi(\omega),\quad G_1(\omega)=\Pi_2(\omega)\Pi(\omega) \tag{1.20}$$
$$\Pi=\prod_1^N\left[\frac{(A_k-\omega^{-1})(\omega-B_k)}{(A_k-\omega)(\omega^{-1}-B_k)}\right]^{\delta_1}$$
$$\Pi_m=\prod_1^N\frac{(\omega+\omega^{-1}-B_k-B_k^{-1})^{(-1)^m/2}}{(A_k+A_k^{-1}-\omega-\omega^{-1})^{1/2}}=$$
$$2^{N(m-2)}\prod_1^N(z-b_k)^{(-1)^m/2}(a_k-z)^{-1/2},\quad \alpha\,\overline{\in}\,[0,1];\quad \Pi_m\equiv1$$
$$\alpha\in\,]0,1[$$
$$F_1=\frac{\omega}{\omega+1},\quad F_1^\circ=\frac{\omega}{\omega^2-1},\quad |\alpha|<\infty;\quad R_1=ir_0=i\bar{r}_0 \tag{1.21}$$
$$0<\alpha<1;$$
$$R_1=\sum_{-N}^{N}r_k\omega^k;\quad r_k=-\bar{r}_{-k}\quad \alpha>1;\quad r_k=\bar{r}_k,\quad \alpha<0$$

Assuming, for simplicity, that the quantities $f(x)$ decrease no slower at infinity than $1/x$, we can establish the existence of the integral in (1.15) and that the solution decreases as $1/z$ as $z\to\infty$.

The total index of the initial problem $\varkappa$ equals the sum of the orders of the singularities of the canonical solutions at all the singularities $z=a_k, b_k, \pm1$ /2/ in this case

$$\varkappa=2N+1,\ \alpha\,\overline{\in}\,[0,1];\ \varkappa=1,\ \alpha\in\,]0,1[$$

and equals the number of free real constants in the solution (1.15). This is in agreement with the general assertions about the number of linearly independent solutions of the Riemann-Hilbert problem /2/.

2) The interval l_0 delimits both l_1 and l_2. Let one of the points ± 1 be the common boundary point of l_0 and l_1, while the other point ∓ 1 is the common boundary point for l_0 and l_2, i.e., either $a_q = +1$ or $b_p = -1$, where p, q take one of the values $1, 2, \ldots, N$ (the upper or lower signs are taken everyhere). The quantities

$$\Pi_2 = \prod_1^N \left(\frac{A_k + A_k^{-1} - \omega - \omega^{-1}}{\omega + \omega^{-1} - B_k - B_k^{-1}}\right)^{\mp 1/2}, \quad F_1 = F_1^\circ = \frac{\omega}{\omega \pm 1} \tag{1.22}$$

$$R_1 = \sum_{-N}^{N-1} r_k \omega^k, \quad \alpha \overline{\in} [0, 1]; \quad r_k = \pm \bar{r}_{-1-k}, \quad k = 0, \ldots, N-1$$

$$\alpha > 1;$$

$$r_k = \bar{r}_k, \quad k = -N, \ldots, N-1, \quad \alpha < 0$$

are subject to change in (1.20) and (1.21).

Since the functions $\Pi_m(\omega)$ acquired a pole at one of the points ± 1, the functions $F_m(\omega)$ should have lost it.

The index of the problem was decreased by one in the case $\alpha \overline{\in} [0, 1]$ as compared with the modification 1), and the behaviour at infinity remained as before.

3) Discontinuity of the boundary conditions at infinity; l_0 delimits only l_2.

Let $a_1 = A_1 = -\infty$, $a_k \neq 1$, $b_k \neq -1$ and let the behaviour of the solution at infinity be irregular. Constructing auxiliary functions anew (the result can also be obtained by a special passage to the limit $A_1 \to -\infty$ in (1.20)), we see that the expressions for the functions $\Pi(\omega), \Pi_m(\omega)$ differ from (1.20) in the absence of those factors in which the quantity A_1 occurred, while (1.21) remain valid. At infinity

$$W_m(\omega) \sim \omega^{-(1/2+\delta_m)}, \quad \alpha \overline{\in} [0,1]; \quad W_m(\omega) \sim \omega^{-(1+\delta_m)}, \quad \alpha \in]0, 1[$$

If the vanishing of the solution no more slowly than $|z|^{-1}$ is required, then the number of linearly-independent solutions is reduced by two ($\alpha \overline{\in} [0, 1]$) and summation in (1.21) should be performed between $-N+1$ and $N-1$.

The cases $|a_1| < \infty$, $b_N = \infty$ and $|a_1| = b_N = \infty$ are contained analogously. The appropriate cofactors are discarded from the products (1.20) and the limits of the summation in (1.21) are selected, as usual, depending on the behaviour of the solution at infinity.

4) Discontinuity in the boundary conditions at infinity; l_0 delimits both l_1 and l_2. The passage 3) $\to$ 4) is analogous to the passage 1) $\to$ 2).

The solution of problem (1.2) can be constructed by the same method in other function classes, for instance, those bounded near certain points a_k, b_k (the solvability question is examined analogously /2, 8/). The result can be extracted directly from the solution obtained for the class h_0 by letting the coefficients in the appropriate asymptotic forms go to zero.

2. We will consider certain problems on the deformation of a composite elastic plane that converge to problem (1.1), (1.2). The straight line separating the elastic properties $y = 0$ is divided into a system of intervals l_n (see Sect.1) on which conditions of three kinds are posed: any combination of conditions 1°–3° on l_1 and l_2, and one of the conditions 4° or 5° on l_0.

1°. Normal and tangential stresses

$$(\sigma_y + i\tau_{xy})(x, \pm 0) = g^\pm(x) \equiv (-p + i\tau)^\pm(x)$$

are applied to the slot edges.

2°. The displacement vector

$$(u + iv)(x, \pm 0) = h^\pm(x)$$

is given.

3°. Total contact conditions

$$[\sigma_y] = [\tau_{xy}] = [u] = [v] = 0$$

are satisfied.

4°. A shear stress is given and continuity conditions are satisfied for the normal displacement and stress components $\tau_{xy}(x, \pm 0) = \tau^\pm(x)$, $[\sigma_y] = [v] = 0$.

5°. The shear stress and normal displacement are given

$$\tau_{xy}(x, \pm 0) = \tau^\pm(x), \quad v(x, \pm 0) = v^\pm_0(x)$$

Here the square brackets denote a jump in the quantity when passing from the upper to the lower edge.

We consider the stresses to vanish at infinity (a homogeneous stress field at infinity can be expected).

We use complex representations of solutions near to those considered in /9/

$$\sigma_x + \sigma_y = 2\mathrm{Re}\,\{X_2(z) - X_1(z)\} \tag{2.1}$$
$$\sigma_y - i\tau_{xy} = \mathrm{Re}\,\{X_2(z) - X_1(z)\} + \overline{X_1(z)} + iy\,\{\overline{X_2'(z)} - \overline{X_1'(z)}\}$$
$$4\mu_j(u_{,x} + iv_{,x}) = \varkappa_j\{X_2(z) - X_1(z)\} - \overline{X_1(z)} - \overline{X_2(z)} + 2iy\,\{\overline{X_1'(z)} - \overline{X_2'(z)}\}$$
$$4\mu_j(u_{,y} - iv_{,y}) = \varkappa_j\{X_2(z) - X_1(z)\} + 3\overline{X_1(z)} - \overline{X_2(z)} - 2iy\,\{\overline{X_1'(z)} - \overline{X_2'(z)}\}$$

almost everywhere $\lim_{y\to 0} yX_m'(z) = 0,\ m = 1,\ 2;\ X' = dX/dz$.

$$\tau_{xy}(x, \pm 0) = \mathrm{Im}\,X_1^{\pm}(x),\ \sigma_y(x, \pm 0) = \mathrm{Re}\,X_2^{\pm}(x)$$
$$u_{,x}(x, \pm 0) = -\mathrm{Re}\,\{b^{(j)}X_1(x) + a^{(j)}X_2(x)\}^{\pm},$$
$$v_{,x}(x, \pm 0) = \mathrm{Im}\,\{a^{(j)}X_1(x) + b^{(j)}X_2(x)\}^{\pm};\ 4\mu_j a^{(j)} = 1 - \varkappa_j,\ 4\mu_j b^{(j)} = 1 + \varkappa_j$$

The functions $X_m(z)$ are regular in the plane z, except at points of the real axis, have a limit as $y \to \pm 0$ almost everywhere, are subject to the estimates (1.1) near the singular points ($z \neq \infty$) and vanish as $z \to \infty$; the superscript $j = 1$ ($j = 2$) fixes the parameters of the medium occupying the half-plane $y > 0$ ($y < 0$), μ_j are the shear moduli, $\varkappa_j = 3-4\nu_j$ (plane strain), $\varkappa_j = (3 - \nu_j)(1 + \nu_j)^{-1}$ (generalized plane state of stress), ν_j are Poisson's ratios.

3. We will show in more detail the transition from the problem combinations of boundary conditions 1°, 3°, 4° of the bonding line is weakened by a system of slots l_2 and by a transverse shear crack l_0 (l_0 delimits only l_1) or one of the slots goes over to a shear crack (l_0 delimits l_1 and l_2) or the edges of one of the slots makes contact on the section l_0 (l_0 delimits only l_2).

Let loads be applied symmetrically first: $g^+(x) = g^-(x)$. Then by virtue of 1°, 3°, 4° and (2.1) the following continuation conditions hold:

$$X_1(z) = -\overline{X_1(\bar z)},\ X_2(z) = \overline{X_2(\bar z)} \tag{3.1}$$

which enables us to consider the problem in just the upper half-plane. The boundary conditions take the form (1.2) if the load on l_0 is reduced first by subtracting the solution of an auxiliary problem. Assuming $\tau^{\pm}(x) = 0,\ x \in l_0$ for simplicity, we arrive at problem (1.1)–(1.3), where

$$d_{11} = d_{22} = d,\ d_{12} = d_{21} = q,\ d = a^{(1)} - a^{(2)} \tag{3.2}$$
$$q = b^{(1)} + b^{(2)};\ f(x) = g^+(x),\ x \in l_2;\ f(x) = 0,\ x \in l_1$$
$$\alpha = q^2 d^{-2} > 1,\ \lambda_1 = (d - q)/(d + q) < 0\ \ (s = -1)$$

If the shear crack is isolated, then by a linear substitution of functions the problem is reduced to modifications examined in Sect.1. However, it is simpler to use the results (1.20), (1.21) directly by taking into account that in this case the auxiliary function $\Pi_1(\omega)$ has a pole at the points $\omega = \pm 1$, while the function $\Pi_2(\omega)$ has the pole at the point $\omega = 1$. The expression for $F^\circ(\omega)$ changes: $F^\circ(\omega) = 1$. The principal vector of the forces applied to the boundaries equals zero, and $X_m(z) = O(1/z^2)$ as $z \to \infty$ /10/. For $a_1^2 + b_N^2 < \infty$ (the bonding sections have a finite perimeter) the summation in (1.21) should be performed from $1 - N$ to $N - 1$. If $a_1 = -\infty,\ b_N = \infty$ (all the slots have finite length), then the factors containing a_1 and b_N are discarded from the products in (1.20) and the limits of the summation in (1.21) are $2 - N$ and $N - 2$. Other modifications are analogous to case 3) from Sect.1.

Let l_0 delimit just l_2 (or l_2 and l_1). In this case the formulation of the problem with contact boundaries not known in advance is physically meaningful (the ends l_0^*, where l_n^* are the corresponding sections in the laboratory system of coordinates). This additional complication of the problem is overcome by the usual means. Determining l_0^* in a certain manner, we execute the coordinate system transformation for which $l_0^* \to l_0$. The unknown coordinates of the boundaries l_0^* enter the subsequent formulas as parameters. Additional conditions in the form of the inequalities

$$\sigma_y - \sigma_y^\circ \leqslant 0,\ x \in l_0;\ [v] + \delta^\circ \geqslant 0,\ x \in l_2$$

moreover occur in the problem, where σ_y° is the reduced normal stress and δ° is the initial slot gap. Continuity of the stresses at the unkown points follows from these inequalities. Equating the corresponding stress intensity factors to zero, we obtain an equation to determine the coordinates of these points. The unique roots are selected in verifying the inequalities mentioned /7/.

The complex constants r_k are determined from the condition of single-valuedness of the displacements when taking account of the relations metnioned in (1.21) similar to /6, 10/.

Thus in the case of an isolated shear crack and slots having finite perimeter, we obtain two conditions for the single-valuedness of the components u and v for traversal around each slot, and one condition for the single-valuedness of the component u for traversal around the shear crack. In all we have $2N-3$ independent equations to determine $2N-3$ real free constants. The question of the uniqueness of the solution is thereby exhausted. We note that the uniqueness of the solutions of certain problems of elasticity theory with indeterminate points of boundary condition separation is established in /11/.

When there is no symmetry $(g^+(x) \neq g^-(x))$ we consider first the following auxiliary problem for the vector components $\vartheta = (\vartheta_1, \vartheta_2)$ in order to reduce the inhomogeneities in the boundary conditions at l_0 and l_2 $(\vartheta(z) \to 0,\ z \to \infty)$

$$\operatorname{Im} \vartheta_1^{\pm}|_{l_0+l_2} = \tau^{\pm}(x),\ \operatorname{Im} \vartheta_1^{\pm}|_{l_1} = 0,\quad \operatorname{Re} \vartheta_2^{\pm}|_{l_1} = -p^{\pm}(x)$$
$$\operatorname{Re} \vartheta_2^{\pm}|_{l_0+l_2} = 0$$

The solutions of these problems can be expressed simply in terms of Cauchy-type integrals /10/.

For a piecewise-holomorphic vector $\Psi(z) = \mathrm{X}(z) - \vartheta(z)$ the continuity conditions $\operatorname{Im} \Psi_1^+ = \operatorname{Im} \Psi_1^-$ and $\operatorname{Re} \Psi_2^+ = \operatorname{Re} \Psi_2^-$ will be satisfied on the whole real axis, and therefore, continuation conditions analogous to (3.1) will hold. Consequently, we obtain the problem

$$\begin{aligned} &\operatorname{Im} \Psi_1^+ = 0,\ \operatorname{Im} \Psi_2^+ = f_0(x)\ \ (l_0) \\ &\operatorname{Im} (D_1\Psi^+) = f(x)\ \ (l_1),\ \operatorname{Im} \Psi_1^+ = \operatorname{Re} \Psi_2^+ = 0\ \ (l_2) \end{aligned} \tag{3.3}$$

to determine the vector $\Psi(z)$, analytic in the upper half-plane, where the matrix D_1 is determined in (1.3), (3.2) and $f_0(x), f(x)$ are expressed in terms of the boundary values of the functions $\vartheta_m(z)$.

For the final reduction of the problem to the form (1.2), the inhomogeneity in the boundary condition must be reduced by l_0 from (3.3). This can be done by forming the vector $\Phi = (\Psi_1, \Psi_2 - \Psi_0)$, where

$$\Psi_0 = \frac{i}{\pi\sqrt{z^2-1}} \int_{-1}^{1} \frac{f_0(t)\sqrt{1-t^2}}{t-z}\,dt,\quad \operatorname{Im} \Psi_0^+|_{l_0} = f_0(x)]$$
$$\operatorname{Re} \Psi_0^+|_{l_1+l_2} = 0$$

The principal vector of the external forces (X, Y) can be differeent from zero. The asymptotic form of the solution at infinity is determined by the formulas /5/

$$\begin{aligned} &\mathrm{X}_k(z) = \frac{\gamma_{kj}X - i\gamma_{mj}Y}{\pi z S} + O\left(\frac{1}{z^2}\right)\quad (j,\ k,\ m = 1,\ 2;\ k \neq m) \\ &\Phi_1 \sim \frac{\gamma_{1j}X - i\gamma_{2j}Y \pm ST^{\pm}}{\pi z S},\quad \Phi_2 \sim \frac{\gamma_{2j}X - i(\gamma_{1j}Y \mp SP^{\pm})}{\pi z S} \\ &y \to \pm\infty \\ &X = T^- - T^+,\ \ Y = P^+ - P^-,\ \ S = d^2 - g^2 \\ &T^{\pm} = \int_{l_0+l_2} \tau^{\pm}(x)\,dx,\quad P^{\pm} = \int_{l_1} p^{\pm}(x)\,dx \\ &\gamma_{11} = -a^{(2)}d - b^{(2)}g,\quad \gamma_{12} = a^{(1)}d - b^{(1)}g \\ &\gamma_{22} = \gamma_{21} = a^{(1)}b^{(2)} + a^{(2)}b^{(1)} \end{aligned} \tag{3.4}$$

The upper signs are taken and $j = 1$ for $y > 0$, and the lower signs and $j = 2$ for $y < 0$. It can be confirmed that conditions (3.1) are satisfied for the asymptotic forms $\Phi_1(z)$ and $\Phi_2(z)$.

Compared with the previous case, the solution has a weaker decrease at infinity, the number of real constants to be determined has increased by two. But even the number of independent conditions, including the previous conditions for single-valuedness of the displacements during traversal of the contours l_0, l_2 and the elements (3.4) grew just as much (we recall that conditions (3.4) include the conditions for single-valuedness of the displacement vector during traversal of the contour enclosing the whole interval $l_0 + l_2$ /5, 10/).

Remark. If the shear stress is not given explicitly on the section l_0, but is expressed in terms of another unknown function (for instance, a dry friction condition is posed), then the representations obtained can be used to derive the integral equation of the problem for one unknown function.

Other problems are examined analogously. The cleavage case (without friction) for one of the slots by a rigid insert with a given contact zone corresponds to this combination of conditions 1°,3°,5°. The set of conditions 2°, 3°, 4° (or 5°) corresponds to the problem of a system of sealed-in thin stiff inclusions and one delamination section. In solving this problem it is more convenient to go over to other complex representations

$$\Psi = A^{(j)}X(z),\quad A^{(j)} = \{a_{km}^{(j)}\},\quad a_{11}^{(j)} = -a_{22}^{(j)} = a^{(j)},\quad a_{12}^{(j)} = -a_{21}^{(j)} = b^{(j)}$$
$$u_{,x}(x, \pm 0) = \operatorname{Re}\Psi_2^{\pm}(x),\quad v_{,x}(x, \pm 0) = \operatorname{Im}\Psi_1^{\pm}(x)$$

A closed-form solution can also be obtained in studying the question of contact (with slip) of two half-planes loaded by stresses at l_1, by displacements $(1°, 2°, 4°)$ at l_2, and in the problem of the action of a system of stamps that adhere to a half-plane and one stamp with slip in the half-plane $(1°, 2°, 5°)$.

We limit ourselves to a remark concerning the solution of the above-mentioned problems with defects on the interface under the action of just a homogeneous stress field at infinity. In this case it is best to seek the solution under the condition at infinity $(\beta = -d/q)$

$$X_1 = \beta\sigma_y^\circ + i\tau_{xy}^\circ + O(z^{-2}),\quad X_2 = \sigma_y^\circ + i\beta\tau_{xy}^\circ + O(z^{-2}),\quad y > 0 \tag{3.5}$$

The principal part of the asymptotic form (3.5) is the homogeneous solution for a composite elastic plane without defects (we take condition (3.1) into account)

$$\sigma_y = \sigma_y^\circ,\quad \tau_{xy} = \tau_{xy}^\circ,\quad |y| \leqslant \infty$$
$$\sigma_x = (1 - 2\beta)\sigma_y^\circ,\quad y > 0,\quad \sigma_x = (1 + 2\beta)\sigma_y^\circ,\quad y < 0$$

The solution (1.11), (1.15) will not contain integrals but in the summation limits the N should be replaced by $N+1$ (formulas (1.21) and (1.22)). Condition (3.5) is used to determine the excess constants.

REFERENCES

1. SIMINOV I.V., On the steady motion of a crack with slip and separation sections on the interface of two elastic materials, PMM, 48, 3, 1984.
2. VEKUA N.P., Systems of Singular Integral Equations and Certain Boundary Value Problems. Nauka, Moscow, 1970.
3. GAKHOV F.D., The Riemann boundary value problem for systems of n pairs of functions, Uspekhi Matem. Nauk, 7, 4, 1952.
4. CHEBOTAREV G.N., On the closed-form solution of a Riemann boundary value problem for systems of n pairs of functions, Uchen. Zapisk. Kazan. Univ., 116, Book 4, 1956.
5. CHEREPANOV G.P., The Riemann-Hilbert problem for the exterior of slits along a line or along a circle. Dokl. Akad. Nauk SSSR, 156, 2, 1964.
6. CHEREPANOV G.P., On the state of stress in an inhomogeneous plate with slits, Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, Mekhan. Mashinostr. 1, 1962.
7. NAKHMEIN E.L., NULLER B.M. and RYVKIN M.B., Deformation of a composite elastic plane weekend by a periodic system of arbitrarily loaded slots, PMM, 45, 6, 1981.
8. MUSKHELISHVILI N.I., Singular Integral Equations, Nauka, Moscow, 1968.
9. GALIN L.A., The impression of a stamp in the presence of friction and adhesion, PMM, 9, 5, 1945.
10. MUSKHELISHVILI N.I., Certain Fundamental Problems of the Mathematical Theory of Elasticity, Nauka, Moscow, 1966.
11. SHIELD R.T., Uniqueness for elastic crack and punch problems, Trans. ASME, J. Appl. Mech., 49, 3, 1982.

Translated by M.D.F.